

Representations of the Poincaré group on relativistic phase space

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Abstract

We introduce a complex relativistic phase space as the space \mathbb{C}^4 equipped with the Minkowski metric and with a geometric tri-product on it. The geometric tri-product is similar to the triple product of the bounded symmetric domain of type IV in Cartan's classification, called the spin domain. We define a spin 1 representations of the Lie algebra of the Poincaré group by natural operators of this tri-product on the complex relativistic phase space. This representation is connected with the electromagnetic tensor. A spin 1/2 representation on the complex relativistic phase space is constructed by use of the complex Faraday electromagnetic tensor. We show that the Newman-Penrose basis for the phase space determines the Dirac bi-spinors under this representation. Quite remarkable that the tri-product representation admits only spin 1 and spin 1/2 representations which correspond to most particles of nature.

Keywords: Complex relativistic phase space, Poincaré group, Dirac bi-spinors, Geometric tri-product, spin domain.

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1 Introduction

The spin factor, a bounded symmetric domain of type IV in the Cartan classification [3], can play an important role in physics. It was shown in [8] that the state space of any two-state quantum system is the dual of a complex spin factor and the geometry of the state space can be defined by the triple product of the spin domain. In [5] and [10] it was shown that a new dynamic variable, called s -velocity, which is a relativistic half of the usual velocity, is useful for solving explicitly relativistic dynamic equations. It was shown that the automorphism group, generated by s -velocity addition, coincides

with the conformal group. The Lie algebra of this group is described by the triple product defined by the spin domain.

In [6] we introduced a geometric tri-product as a generalization of the geometric product in Clifford algebras. This product coincides with the triple product of the spin domain. In this paper, in order to make this mathematical model closer to the physical reality, we modify this tri-product and define a relativistic phase space as follows. We equip the space \mathbb{C}^4 with an inner product based on the Lorentz metric and define a new geometric tri-product on it. This space is used to represent both the space-time coordinates and the four-momentum of an object. The real part of the inner product extends the notion of the Lorentz scalar product on the space-time and energy-momentum, while the imaginary part extends the symplectic skew scalar product of the classical phase space.

We construct both spin 1 and spin 1/2 representations of the Poincaré group by natural operators of the tri-product on the phase space. The generators of space-time translations of the Lie algebra of the Poincaré group are represented by the basic vectors of the relativistic phase space while the generators of relativistic angular momentum are represented by natural operators of the triple product. More precisely, for spin 1 representation, the generators of boosts are presented by operators with the meaning of electric field strength tensors while the generators of rotations are presented by operators with the meaning of magnetic field strength tensors. For the spin 1/2 representation, the generators of boosts and of rotations are presented by the complex Faraday electromagnetic tensor. We show that if we use the Newman-Penrose basis on the phase space, we obtain the Dirac bi-spinors under this representation.

In a forthcoming paper we will describe the significance of the Newman-Penrose basis for the relativistic phase space.

2 Commutation relations of the Poincaré algebra

In this section we recall several known facts about the Lie algebra of the Poincaré group. It is well-known that the laws of physics must be invariant under the following transformations: 1) the space-time translations, 2) the space rotations, and 3) the proper Lorentz transformations (boosts). These transformations generate the Poincaré group.

A basis of the Lie algebra of the Poincaré group consists of generators of space-time translations, denoted by P_μ for $\mu = 0, 1, 2, 3$ (all Greek indices range from 0 to 3) and generators of rotations and boosts, called relativistic angular momentum, denoted by $M_{\alpha\beta}$. For

a scalar particle, described by a wave function on flat space-time with the metric tensor $\eta_{\mu\alpha} = \text{diag}(1, -1, -1, -1)$, these generators act as

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad (1)$$

and

$$M_{\alpha\beta} = X_\alpha P_\beta - X_\beta P_\alpha, \quad (2)$$

where the operator X_α acts as multiplication of the wave function by $x_\alpha = \eta_{\mu\alpha} x^\mu$.

The commutation relations of the Poincaré algebra in flat space with the metric tensor $\eta_{\mu\alpha}$ are given by

$$[P_\mu, P_\nu] = 0, \quad (3)$$

$$[M_{\alpha\beta}, P_\mu] = \eta_{\mu\beta} P_\alpha - \eta_{\mu\alpha} P_\beta, \quad (4)$$

and

$$[M_{\mu\nu}, M_{\alpha\beta}] = \eta_{\mu\beta} M_{\nu\alpha} + \eta_{\nu\alpha} M_{\mu\beta} - \eta_{\mu\alpha} M_{\nu\beta} - \eta_{\nu\beta} M_{\mu\alpha}. \quad (5)$$

Recall also additional commutation relations:

$$[P_\mu, X_\alpha] = \eta_{\mu\alpha} I, \quad (6)$$

where I denotes the identity operator,

$$[X_\alpha, X_\beta] = 0,$$

and

$$[M_{\alpha\beta}, X_\mu] = \eta_{\mu\beta} X_\alpha - \eta_{\mu\alpha} X_\beta. \quad (7)$$

The commutation relations (3)-(5) of the Poincaré algebra show that the real span of its generators $\mathcal{L} = \text{span}_R\{P_\mu, M_{\alpha\beta}\}$ has a structure of a graded Lie algebra. The $\text{span}_R\{M_{\alpha\beta}\}$ is a Lie subalgebra \mathcal{L}_0 corresponding to the grade 0. This is the Lie algebra of the Lorentz group as a subgroup of the Poincaré group. The bracket of elements of \mathcal{L}_0 with P_μ belong to $\text{span}_R\{P_\mu\}$. So, $\mathcal{L}_1 = \text{span}\{P_\mu\}$ is the grade 1 of the algebra. Commutation relations (3) imply that the brackets on \mathcal{L}_1 are trivial. So

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1, \quad \mathcal{L}_0 = \text{span}_R\{M_{\alpha\beta}\}, \quad \mathcal{L}_1 = \text{span}_R\{P_\mu\} \quad (8)$$

is a graded Lie algebra with a representation of the Poincaré algebra on it.

Our aim is to find a spin 1 and spin 1/2 representations of the Poincaré algebra. To do this we will complexify \mathcal{L} by introducing a complex relativistic phase space with a tri-product on it.

3 A complex relativistic phase space

For description of classical motion of a point-like body, the classical phase space, composed from space position and the 3-momentum, is used. A symplectic structure on the phase space provides information on the scalar product and the antisymmetric symplectic form of this space. For the classical phase space this structure can be expressed efficiently by introducing the complex structure and a scalar product under which the classical phase space becomes equal to \mathbb{C}^3 , see [12] and [1].

For description of the motion of a *relativistic* point-like body we may use the relativistic phase space. This is obtained by adding the time and the energy variables to the classical phase space. So, the relativistic phase space may be identified with \mathbb{C}^4 , in which the real part expresses the 4-momentum and the imaginary part represents the space-time position of a point-like body.

To extend the symplectic structure to this space, we will use the complex-valued scalar product introduced by E. Cartan, see [4]. Since the scalar product needs to provide information on the interval of the 4-vectors, we replace the Euclidian metric, used usually, with the Lorentzian one. We chose the scalar product to be complex linear in both terms and not, as usually, conjugate linear in one term. This is consistent with the fact that the Lorentz invariant of an electromagnetic field is given by $\mathbf{F}^2 = (\mathbf{E} + i\mathbf{B})^2$ and not by $|\mathbf{F}|$.

Since at any given point the action of generators of space-time translations P_μ could be identified with four vectors in the tangent space, we will present them by basic real vectors $\{\mathbf{u}_\mu\}$ in \mathbb{C}^4 . We define a scalar product $\langle \cdot | \cdot \rangle$ on \mathbb{C}^4 as follows:

Definition 3.1 *On the real basis vectors of \mathbb{C}^4 a scalar product $\langle \cdot | \cdot \rangle$ is given by*

$$\langle \mathbf{u}_\mu | \mathbf{u}_\nu \rangle = \eta_{\mu\nu}. \quad (9)$$

For two arbitrary vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^4$ it is given by

$$\langle \mathbf{a} | \mathbf{b} \rangle = \langle a^\mu \mathbf{u}_\mu | b^\nu \mathbf{u}_\nu \rangle = \eta_{\mu\nu} a^\mu b^\nu, \quad (10)$$

where $\mathbf{a} = a^\mu \mathbf{u}_\mu$ and $\mathbf{b} = b^\mu \mathbf{u}_\mu$.

Evidently, this scalar product is bilinear and symmetric. For an arbitrary element $\mathbf{a} \in \mathbb{C}^4$, the *scalar square* is given by

$$\mathbf{a}^2 = \langle \mathbf{a} | \mathbf{a} \rangle = \eta_{\mu\nu} a^\mu a^\nu, \quad (11)$$

which is a complex number, not necessary positive or even real.

Note that the real subspace M defined by the real vectors of \mathbb{C}^4 with the bilinear form (11) may be identified with the Minkowski space. The same is true for the pure imaginary subspace iM defined by pure imaginary vectors. We use the subspace M to represent the

four-vector momentum $\mathbf{p} = p^\mu \mathbf{u}_\mu$ and the subspace iM to represent the space-time coordinates $i\mathbf{x} = x^\mu i\mathbf{u}_\mu$ of a point-like body in an inertial system. Thus, the space \mathbb{C}^4 represents both space-time and the four-momentum as

$$\mathbf{a} = a^\mu \mathbf{u}_\mu \text{ with } a^\mu = p^\mu + ix^\mu.$$

The square of elements of M are \mathbf{p}^2 which is known to be equal to $(m_0 c)^2$ with m_0 the rest mass of the object, while the negative of the square $-(i\mathbf{x})^2$ of elements of iM have the meaning of space-time interval. Note that both $\mathbf{a}^2 = \mathbf{p}^2 - \mathbf{x}^2 + 2i \langle \mathbf{p} | \mathbf{x} \rangle$ and $\langle \bar{\mathbf{a}} | \mathbf{a} \rangle = \mathbf{p}^2 + \mathbf{x}^2$ (where $\bar{\mathbf{a}} = \bar{a}^\mu \mathbf{u}_\mu$ denotes the complex conjugate of \mathbf{a}) are Lorentz invariants and that

$$\mathbf{p}^2 = \frac{1}{2} Re(\langle \bar{\mathbf{a}} | \mathbf{a} \rangle + \mathbf{a}^2), \quad \mathbf{x}^2 = \frac{1}{2} Re(\langle \bar{\mathbf{a}} | \mathbf{a} \rangle - \mathbf{a}^2).$$

For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^4$ we may consider also a scalar product $\langle \bar{\mathbf{a}} | \mathbf{b} \rangle = \eta_{\mu\nu} \bar{a}^\mu b^\nu$. The Lorentzian scalar product is the real part

$$Re \langle \bar{\mathbf{a}} | \mathbf{b} \rangle = \frac{1}{2} \eta_{\mu\nu} (\bar{a}^\mu b^\nu + a^\mu \bar{b}^\nu), \quad (12)$$

which is symmetric and extends the Lorentzian product (and the notion of an interval) from the subspaces M and iM to $\mathbb{C}^4 = M \oplus iM$.

The imaginary part defines a skew scalar product

$$[\mathbf{a}, \mathbf{b}] = Im \langle \bar{\mathbf{a}} | \mathbf{b} \rangle = \frac{1}{2i} \eta_{\mu\nu} (\bar{a}^\mu b^\nu - a^\mu \bar{b}^\nu), \quad (13)$$

which extends the symplectic skew scalar product. This bracket can be used to define the Poisson bracket of two functions and two vector fields. Thus, the space \mathbb{C}^4 with the scalar product (10) can be used as a basis for a relativistic phase space.

The commutation relations (4) suggest that the relativistic angular momentum $M_{\alpha\beta}$ can be presented as operators on the space \mathbb{C}^4 . Moreover, such operators will be bilinear in $\mathbf{u}_\alpha, \mathbf{u}_\beta$ and antisymmetric in these variables. The commutation relation could be used to define a triple product of $\mathbf{u}_\alpha, \mathbf{u}_\beta, \mathbf{u}_\mu$.

Definition 3.2 Let \mathbb{C}^4 denote a 4-dimensional complex space with the scalar product (10). A **geometric tri-product** $\{ , , \} : \mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C}^4$ is defined for any triple of elements \mathbf{a}, \mathbf{b} and \mathbf{c} as

$$\mathbf{d} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \langle \mathbf{a} | \mathbf{b} \rangle \mathbf{c} - \langle \mathbf{c} | \mathbf{a} \rangle \mathbf{b} + \langle \mathbf{b} | \mathbf{c} \rangle \mathbf{a}. \quad (14)$$

In the basis $\{\mathbf{u}_\mu\}$ definition (14) takes the form

$$d^\mu = \eta_{\alpha\beta} a^\alpha b^\beta c^\mu - \eta_{\alpha\beta} c^\alpha a^\beta b^\mu + \eta_{\alpha\beta} b^\alpha c^\beta a^\mu. \quad (15)$$

Definition 3.3 For any pair of elements $\mathbf{a}, \mathbf{b} \in \mathbb{C}^4$ we define a linear map $D(\mathbf{a}, \mathbf{b}) : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ as

$$D(\mathbf{a}, \mathbf{b})\mathbf{c} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \quad (16)$$

and an antisymmetric map $\hat{D}(\mathbf{a}, \mathbf{b})$ as

$$\hat{D}(\mathbf{a}, \mathbf{b}) = \frac{1}{2}(D(\mathbf{a}, \mathbf{b}) - D(\mathbf{b}, \mathbf{a})). \quad (17)$$

It is easy to verify that the geometric tri-product satisfies the following properties:

Proposition 3.1 The tri-product, defined by (14), satisfies:

1. $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is complex linear in all variables \mathbf{a} , \mathbf{b} and \mathbf{c} .
2. The triple product is symmetric in the pair of outer variables

$$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{\mathbf{c}, \mathbf{b}, \mathbf{a}\}. \quad (18)$$

3. For arbitrary $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} \in \mathbb{C}^4$, the following identity holds

$$[D(\mathbf{x}, \mathbf{y}), D(\mathbf{a}, \mathbf{b})] = D(D(\mathbf{x}, \mathbf{y})\mathbf{a}, \mathbf{b}) - D(\mathbf{a}, D(\mathbf{y}, \mathbf{x})\mathbf{b}). \quad (19)$$

Properties of the previous proposition are the defining properties for the Jordan triple products associated with a homogeneous spaces, see [13] and [5]. If the Euclidean inner product of \mathbb{C}^4 is used in the definition (14), this triple product is the triple product of the bounded symmetric domain of type IV in Cartan's classification, called the spin factor. A similar triple product was obtained [5] for the ball of relativistically admissible velocities under the action of the conformal group. As we will see later, the geometric tri-product (14) is useful in defining the action of the Lorentz group on \mathbb{C}^4 .

The space \mathbb{C}^4 with a form (10) for the given metric tensor and a geometric tri-product (14) will be denoted by \mathcal{S}^4 . As we have seen, the space \mathcal{S}^4 can be used to represent the space-time coordinates and the relativistic momentum variables. The form (10) on it defines both the interval and the symplectic form and the tri-product may be used to define the action of the Lorentz group. Thus, we propose to call \mathcal{S}^4 the *complex relativistic phase space*.

4 The quasi-orthogonal group $QO(\mathcal{S}^4)$ and its Lie algebra $qo(\mathcal{S}^4)$

As in [11] we define:

Definition 7.1 An invertible linear map $T \in GL(\mathcal{S}^4)$ which preserves the scalar product (10) will be called a *quasi-orthogonal* map. The

group of all quasi-orthogonal maps on \mathcal{S}^4 will be called the *quasi-orthogonal group* and denoted by $QO(\mathcal{S}^4)$. The Lie algebra of $QO(\mathcal{S}^4)$ will be denoted by $qo(\mathcal{S}^4)$.

The Lorentz group, preserving the intervals, also preserves the scalar product (10). Thus, $QO(\mathcal{S}^4)$ can be identified with the Lorentz group and its Lie algebra $qo(\mathcal{S}^4)$ with the Lorentz algebra.

From the definition of $QO(\mathcal{S}^4)$ we have

$$QO(\mathcal{S}^4) = \{g \in GL(\mathcal{S}^4) : \langle g\mathbf{a}|g\mathbf{b} \rangle = \langle \mathbf{a}|\mathbf{b} \rangle, \mathbf{a}, \mathbf{b} \in \mathcal{S}^4\}. \quad (20)$$

If $g(t)$ is a smooth curve in $QO(\mathcal{S}^4)$, with $g(0) = I$, the identity map on \mathcal{S}^4 , then $X := g'(0) \in qo(\mathcal{S}^4)$. Since $g(t) \in QO(\mathcal{S}^4)$, from (20) we have

$$\langle g(t)\mathbf{a}|g(t)\mathbf{b} \rangle = \langle \mathbf{a}|\mathbf{b} \rangle.$$

Differentiating this by t and substituting $t = 0$, we obtain

$$\langle X\mathbf{a}|\mathbf{b} \rangle + \langle \mathbf{a}|X\mathbf{b} \rangle = 0. \quad (21)$$

In basis $\{\mathbf{u}_\mu\}$ this equation takes the form

$$(\eta_{\gamma\beta}X_\alpha{}^\gamma + \eta_{\gamma\alpha}X_\beta{}^\gamma)a^\alpha b^\beta = 0,$$

where the operator X is represented by the mixed tensor $X_\alpha{}^\gamma$. This mean that the matrix $X_{\alpha\beta} = \eta_{\gamma\beta}X_\alpha{}^\gamma$ is antisymmetric. The space of such antisymmetric two-tensors is a 6-dimensional complex space that is denoted in the literature as \mathcal{M}^6 , see [11].

Using the triple product (14) on \mathcal{S}^4 we see that the linear operator $D_{\alpha\beta} = \hat{D}(\mathbf{u}_\alpha, \mathbf{u}_\beta)$, defined by (17) act on the basis elements as

$$D_{\alpha\beta}\mathbf{u}_\gamma = \hat{D}(\mathbf{u}_\alpha, \mathbf{u}_\beta)\mathbf{u}_\gamma = -\eta_{\gamma\alpha}\mathbf{u}_\beta + \eta_{\beta\gamma}\mathbf{u}_\alpha = -D_{\beta\alpha}\mathbf{u}_\gamma. \quad (22)$$

Thus $D_{\alpha\beta}$ are elements and span the Lie algebra $qo(\mathcal{S}^4)$. Note that for $\alpha \neq \beta$, $D_{\alpha\beta} = D(\mathbf{u}_\alpha, \mathbf{u}_\beta)$. We can express this Lie algebra as

$$qo(\mathcal{S}^4) = \{x^{\alpha\beta}D_{\alpha\beta} : x^{\alpha\beta} \in \mathbb{C}, x^{\beta\alpha} = -x^{\alpha\beta}\}. \quad (23)$$

The Lie bracket of the basis elements of this Lie algebra can be calculated by use of (19) and (22) as

$$[D_{\mu\nu}, D_{\alpha\beta}] = D(D(\mathbf{u}_\mu, \mathbf{u}_\nu)\mathbf{u}_\alpha, \mathbf{u}_\beta) - D(\mathbf{u}_\alpha, D(\mathbf{u}_\nu, \mathbf{u}_\mu)\mathbf{u}_\beta) \quad (24)$$

$$= \eta_{\nu\alpha}D_{\mu\beta} - \eta_{\mu\alpha}D_{\nu\beta} + \eta_{\nu\beta}D_{\alpha\mu} - \eta_{\mu\beta}D_{\alpha\nu},$$

which is similar to the commutation relations (5) of the angular momentum in the Lorentz group.

5 Spin 1 representation of the Poincaré algebra on \mathcal{S}^4

To obtain a representation of the Poincaré algebra on \mathcal{S}^4 we will use the complex extension of the algebra \mathcal{L} defined by (8). The subspace \mathcal{L}_1 becomes the relativistic phase space \mathcal{S}^4 and $\mathcal{L}_0 = qo(\mathcal{S}^4)$. The symplectic structure on $\mathcal{L}_1 = \mathcal{S}^4$ defines a nontrivial bracket which results in a new subspace \mathcal{L}_2 of grade 2 consisting of constants. The brackets of elements of \mathcal{L}_2 with any element of \mathcal{L} are trivial. Thus, we define a graded complex Lie algebra

$$\mathcal{L}(\mathcal{S}^4) = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2, \quad (25)$$

with $\mathcal{L}_1 = \mathcal{S}^4$, $\mathcal{L}_0 = qo(\mathcal{S}^4)$, defined by (23) and $\mathcal{L}_2 = \mathbb{C}$.

The brackets on \mathcal{L}_0 are the usual operator brackets. We define the bracket of any element of $\mathcal{L}(\mathcal{S}^4)$ with any element of \mathcal{L}_2 to be trivial. The bracket of two elements of \mathcal{L}_1 , defined by (13), is constant and can be considered as an element of \mathcal{L}_2 . Finally, the bracket the basis elements of \mathcal{L}_0 and \mathcal{L}_1 , by use of (22), is defined as

$$\begin{aligned} [D_{\alpha\beta}, \mathbf{u}_\mu] &= \frac{1}{2}(\{\mathbf{u}_\alpha, \mathbf{u}_\beta, \mathbf{u}_\mu\} - \{\mathbf{u}_\mu, \mathbf{u}_\alpha, \mathbf{u}_\beta\}) \\ &= \frac{1}{2}(D_{\alpha\beta}\mathbf{u}_\mu - D_{\beta\alpha}\mathbf{u}_\mu) = D_{\alpha\beta}\mathbf{u}_\mu. \end{aligned} \quad (26)$$

Similarly,

$$[\mathbf{u}_\mu, D_{\alpha\beta}] = D_{\beta\alpha}\mathbf{u}_\mu = -D_{\alpha\beta}\mathbf{u}_\mu.$$

We define a representation of the Poincaré algebra in $\mathcal{L}(\mathcal{S}^4)$ by

$$\pi(P_\alpha) = \mathbf{u}_\alpha, \quad \pi(M_{\alpha\beta}) = D_{\alpha\beta}. \quad (27)$$

From (13) it follows that (3) hold. By use of (22) we get

$$[\pi(M_{\alpha\beta}), \pi(P_\mu)] = D_{\alpha\beta}\mathbf{u}_\mu = \eta_{\mu\beta}\mathbf{u}_\alpha - \eta_{\mu\alpha}\mathbf{u}_\beta = \eta_{\mu\beta}\pi(P_\alpha) - \eta_{\mu\alpha}\pi(P_\beta) \quad (28)$$

and (4) holds. By use of (24) we get

$$\begin{aligned} [\pi(M_{\mu\nu}), \pi(M_{\alpha\beta})] &= [D_{\mu\nu}, D_{\alpha\beta}] \\ &= \eta_{\mu\beta}\pi(M_{\nu\alpha}) + \eta_{\nu\alpha}\pi(M_{\mu\beta}) - \eta_{\mu\alpha}\pi(M_{\nu\beta}) - \eta_{\nu\beta}\pi(M_{\mu\alpha}). \end{aligned}$$

which coincides with (5). Thus, π defined by (27) defines a representation of the Poincaré algebra into $\mathcal{L}(\mathcal{S}^4)$.

Under this representation, the generators of the boosts are represented as $\pi(M_{0\beta}) = D_{0\beta}$ for $\beta \in \{1, 2, 3\}$. The representation of

the boosts is given by the exponent of $D_{0\beta}$ on M and iM . Since $D_{0\beta}^3 = D_{0\beta}$, the matrix of the boost in x direction ($\beta = 1$) is

$$\exp(\varphi\pi(M_{01})) = \begin{pmatrix} \cosh \varphi & \sinh \varphi & 0 & 0 \\ \sinh \varphi & \cosh \varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (29)$$

which is the usual *Lorentz four-momentum and space-time transformation* for the boost in the x -direction, where $\tanh \varphi = v/c$, and $\mathbf{v} = (v, 0, 0)$ is the relative velocity between the systems.

Similarly, the generators of the rotation are represented as $\pi(M_{\alpha\beta}) = D_{\alpha\beta}$ for $\alpha, \beta \in \{1, 2, 3\}$. Since in this case $D_{\alpha\beta}^3 = -D_{\alpha\beta}$, their exponent defines the regular rotations on the subspaces M and iM . Thus, the representation π is a spin 1 representation. Note that both subspaces M and iM are invariant under this representation.

It is known that an electric field act as a generator of a boost and a magnetic field as a generator of rotations on the four-momentum of a charged particle. The four momentum is represented by the real part of the space \mathcal{S}^4 . So, any electromagnetic field strength tensor could be identified with an element of the real part of the space $qo(\mathcal{S}^4)$ as $\mathfrak{F} = F^{\alpha\beta} D_{\alpha\beta}$. Under this representation, the electric field is presented by $\mathbf{E} = F^{0j} D_{0j}$ with $j = 1, 2, 3$ and the magnetic field is represented by $\mathbf{B} = B_j \frac{1}{2} \epsilon_{0j}^{\alpha\beta} D_{\alpha\beta} = B^1 D_{23} + B^2 D_{31} + B^3 D_{12}$, with $\epsilon_{0j}^{\mu\nu}$ being the Levi-Civita symbol. The Lorentz group will act properly on the electric and magnetic components of the field. The Lorentz force on a particle with four-momentum \mathbf{p} is given by $[\mathfrak{F}, \mathbf{p}] = \mathfrak{F}(\mathbf{p})$. According [11] the imaginary part of \mathcal{M}^6 can be identified with the excitation of the field.

Sometimes it is useful to represent the electromagnetic field not as the electromagnetic tensor \mathbf{F} in which the electric and magnetic components are linearly independent, but as a complex Faraday vector $\mathbf{F}_c = \mathbf{E} + i\mathbf{B}$ in which both components are complex dependent vectors. As we will show in the next section, this can be obtained by by use of a spin 1/2 representation the Poincaré algebra into $\mathcal{L}(\mathcal{S}^4)$.

6 Spin 1/2 representation of the Poincaré algebra on \mathcal{S}^4

The complex linear space $qo(\mathcal{S}^4)$ is a subspace of the space of operators on \mathcal{S}^4 . From (22) it follows that $(D_{\alpha\beta})^3 = -\eta_{\alpha\alpha}\eta_{\beta\beta}D_{\alpha\beta} \in qo(\mathcal{S}^4)$ for any distinct α, β . Thus, $D_{0\alpha}^3 = D_{0\alpha}$. Such operators are called a tripotents. If $j, k \in \{1, 2, 3\}$, then then $D_{j,k}^3 = -D_{j,k}$ and $iD_{j,k}$ is a tripotent in this case.

Define now $D_{0j}^\perp = \frac{1}{2}\varepsilon_{0j}^{\mu\nu}D_{\mu\nu}$, where $\varepsilon_{0j}^{\mu\nu}$ denotes the Levi-Civita symbol with $\varepsilon^{0123} = 1$. For example $D_{01}^\perp = D_{23}$, $D_{23}^\perp = -D_{01}$. With this definition iD_{0j}^\perp is a tripotent and $iD_{0j}^\perp D_{0j} = D_{0j} iD_{0j}^\perp = 0$.

From this one gets that

$$D_{0j}^\pm = D_{0j} \pm iD_{0j}^\perp = D_{0j} \pm i\frac{1}{2}\varepsilon_{0j}^{\mu\nu}D_{\mu\nu} \quad (30)$$

are tripotents for any $j \in \{1, 2, 3\}$. By direct verification you get

$$(D_{0j}^\pm)^2 = I, \quad D_{0j}^\pm D_{0k}^\pm + D_{0k}^\pm D_{0j}^\pm = 0,$$

which can be rewritten as the *canonical anticommutation relations*

$$\frac{1}{2}(D_{0j}^\pm D_{0k}^\pm + D_{0k}^\pm D_{0j}^\pm) = \delta_{jk}I. \quad (31)$$

We introduce now another representation, which will be denoted by π^+ , of the Poincaré algebra in the graded Lie algebra $\mathcal{L}(\mathcal{S}^4)$, defined by (25). We represent first the the relativistic angular momentum by elements of \mathcal{L}_0 as

$$\pi^+(M_{0j}) = \frac{1}{2}D_{0j}^+, \quad \pi^+(M_{kl}) = \frac{i}{2}\varepsilon_{kl}^{0j}D_{0j}^+ \quad (32)$$

where D_{0j}^+ is defined by (30) and $j, k \in \{1, 2, 3\}$. For example, $\pi^+(M_{01}) = \frac{1}{2}(D_{01} + iD_{23})$ and $\pi^+(M_{23}) = \frac{1}{2}(D_{23} - iD_{01}) = -i\pi^+(M_{01})$. Also for this representation the bracket on \mathcal{L}_0 is the usual operator bracket.

To verify that this is a representation of the Lorentz algebra we have to check that (5) is satisfied. Because of the symmetry in (5), it is enough to check 4 cases of these relations:

$$\begin{aligned} [\pi^+(M_{23}), \pi^+(M_{12})] &= \frac{1}{4}[(-iD_{01} + D_{23}), (-iD_{03} + D_{12})] \\ &= \frac{1}{4}([iD_{01}, iD_{03}] - [D_{23}, iD_{03}] - [iD_{01}, D_{12}] + [D_{23}, D_{12}]) \\ &= \frac{1}{4}(-D_{31} + iD_{02} + iD_{02} - D_{31}) = \frac{1}{2}(iD_{02} - D_{31}) = -\pi^+(M_{31}), \\ [\pi^+(M_{01}), \pi^+(M_{31})] &= [i\pi^+(M_{23}), \pi^+(M_{31})] = i\pi^+(M_{21}) = -\pi^+(M_{03}), \\ [\pi^+(M_{01}), \pi^+(M_{03})] &= [i\pi^+(M_{23}), i\pi^+(M_{12})] \\ &= -[\pi^+(M_{23}), \pi^+(M_{12})] = -\pi^+(M_{31}), \end{aligned}$$

and

$$[\pi^+(M_{01}), \pi^+(M_{23})] = [\pi^+(M_{01}), -i\pi^+(M_{01})] = 0.$$

Thus, (5) is satisfied.

We represent the generators of translation, as for the representation π , by $\pi^+(P_\mu) = \mathbf{u}_\mu \in \mathcal{L}_1$, but we need to modify the bracket between \mathcal{L}_0 and \mathcal{L}_1 (instead of (26)) to be

$$[D_{\alpha\beta}^+, \mathbf{u}_\mu] = (D_{\alpha\beta}^+ + \overline{D_{\alpha\beta}^+})\mathbf{u}_\mu = 2D_{\alpha\beta}\mathbf{u}_\mu.$$

This imply that $[\pi^+(M_{\alpha\beta}), \mathbf{u}_\mu] = D_{\alpha\beta}\mathbf{u}_\mu$ and from (28) follow that commutation relations (4) are satisfied for representation π^+ . This finishes the proof that π^+ is a representation of the Poincaré algebra.

From (31) follows that the operator $\pi^+(M_{jk})$, for $j, k \in \{1, 2, 3\}$, representing angular momentum satisfies $\pi^+(M_{jk})^2 = -\frac{1}{4}I$. Thus, the flow generated by them is

$$\exp(\varphi\pi^+(M_{jk})) = \cos(\frac{1}{2}\varphi)I + \sin(\frac{1}{2}\varphi)\pi^+(M_{jk}),$$

showing that the representation π^+ is a spin 1/2 representation. The operator $\pi^+(M_{0k})$ representing generators of boosts satisfies $\pi^+(M_{jk})^2 = \frac{1}{4}I$ and thus its flow is given by

$$\exp(\varphi\pi^+(M_{0k})) = \cosh(\frac{1}{2}\varphi)I + \sinh(\frac{1}{2}\varphi)\pi^+(M_{0k}).$$

In addition to the representation π^+ we can define also a representation π^- by

$$\pi^-(M_{0j}) = \frac{1}{2}D_{0j}^-, \quad \pi^-(M_{jk}) = \frac{i}{2}\varepsilon_{jk}^{\mu\nu}D_{\mu\nu}^-.$$

This representation is also a spin 1/2 representation of the Poincaré algebra.

Let us choose the Newman-Penrose basis $(\mathbf{l}, \mathbf{m}, \mathbf{n}, \overline{\mathbf{m}})$, defined in \mathcal{S}^4 by (see [14])

$$\begin{aligned} \mathbf{l} &= \frac{1}{\sqrt{2}}(\mathbf{u}_0 + \mathbf{u}_3), \quad \mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{u}_1 + i\mathbf{u}_2) \\ \overline{\mathbf{m}} &= \frac{1}{\sqrt{2}}(\mathbf{u}_1 - i\mathbf{u}_2), \quad \mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{u}_0 - \mathbf{u}_3). \end{aligned} \quad (33)$$

Direct calculation show that in this basis the matrices of the generators of boosts M_{0j} and the generators of rotations $J_j = \frac{1}{2}\varepsilon_j^{kl}M_{kl}$ will have a block-matrix form

$$\pi^+(M_{0j}) = -\frac{1}{2} \begin{pmatrix} \bar{\sigma}_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \quad \pi^+(J_j) = \frac{i}{2} \begin{pmatrix} \bar{\sigma}_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \quad (34)$$

in which σ_j are the Pauli matrices. Thus, under this representation, the relativistic phase space \mathcal{S}^4 with the NP basis on it transforms as the Dirac bi-spinors.

As in the previous section, we use the connection of the electromagnetic field with relativistic angular momentum to represent this field tensor by an elements of $\mathcal{L}_0 = qo(\mathcal{S}^4)$. We use now the complex Faraday vector $\mathbf{F}_c = \mathbf{E} + i\mathbf{B}$ to represent the field and represent it by use of the representation π^+ as a *complex Faraday tensor* defined as

$$\mathfrak{F}_c = F_c^j \pi^+(M_{0j}). \quad (35)$$

A similar complex Faraday tensor was introduced along ago by L. Silberstein [15] and was used later in [16]. Note that the electromagnetic and the complex Faraday tensor are related as $\mathfrak{F} = \mathfrak{F}_c + \bar{\mathfrak{F}}_c$.

This representation could be useful for example in calculating the evolution of momentum $\mathbf{p}(\tau)$ a charged particle in a uniform electromagnetic field. The evolution equation is $\frac{d\mathbf{p}(\tau)}{d\tau} = \mathfrak{F}\mathbf{p}(\tau)$, which can be rewritten as $\frac{d\mathbf{p}(\tau)}{d\tau} = (\mathfrak{F}_c + \bar{\mathfrak{F}}_c)\mathbf{p}(\tau)$. Using the fact that the operators \mathfrak{F}_c and $\bar{\mathfrak{F}}_c$ commute, the solution of the evolution equation is

$$\mathbf{p}(\tau) = e^{\mathfrak{F}\tau} \mathbf{p}_0 = e^{\bar{\mathfrak{F}}_c\tau} e^{\mathfrak{F}_c\tau} \mathbf{p}_0,$$

where $\mathbf{p}_0 = \mathbf{p}(0)$.

We denote the complex Lorentz invariant associated with the electro-magnetic field by $z(F) = \mathbf{F}_c^2$. From (31) follows that $\mathfrak{F}_c^2 = (\frac{\sum (F_c^j)^2}{4})I = \frac{z(F)}{4}I$. Define now a complex number $w^2 = \frac{z(F)}{4}$. With this definition $\mathfrak{F}_c^2 = w^2I$ and

$$e^{\mathfrak{F}_c\tau} = \sum \mathfrak{F}_c^n \frac{\tau^n}{n!} = \cosh(w\tau)I + \frac{\sinh(w\tau)}{w}\mathfrak{F}_c.$$

This give an explicit solution of the evolution equation.

7 Conclusions and Discussions

In this paper we introduced a complex relativistic phase space \mathcal{S}^4 as the space \mathbb{C}^4 with a scalar product (10), based on the relativistic metric tensor, and with a geometric tri-product (14) on it. We have seen that the space \mathcal{S}^4 can be used to represent the space-time coordinates and the relativistic momentum variables. The scalar product defines both the Lorentz scalar product and a relativistic extension of the symplectic form. We have shown that the Lorentz algebra is represented by natural operators of the tri-product.

We constructed both spin 1 (in chapter 5) and spin 1/2 (in chapter 6) representations of the Poincaré group by natural operators of the tri-product on the phase space. For the spin 1 representation, the generators of boosts are presented by operators with the meaning of electric field strength tensors while the generators of rotations are presented by operators with the meaning of magnetic field strength

tensors. For the spin $1/2$ representation, the generators of boosts and of rotations are presented by the complex Faraday electromagnetic tensor. We have shown that if we use the Newman-Penrose basis on the phase space, we obtain the Dirac bi-spinors under the spin $1/2$ representation.

We want to propose an explanation of the fact that use of the real electromagnetic tensor for representation of the relativistic angular momentum led to a spin 1 representation, while the use of the complex Faraday tensor led to a spin $1/2$ representation and the Dirac bi-spinors.

The last representation is somehow related with an action of an electro-magnetic field on an electron. An electron, in addition of being a charged particle, has a magnetic momentum, called the spin. A real electromagnetic tensor account only on action of a field on the charged particle, but do not take in account the spin precession. We conject that the complex Faraday tensor describe the full action of the electromagnetic field on the electron. This may explain why the solution of the evolution equation of a charge in a constant uniform field is significantly simpler if we use the complex Faraday tensor instead of the electromagnetic tensor.

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